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Singular invariant equation for the (1 + 1) Fokker–Planck equation

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Abstract

Semi-invariants for the linear parabolic equations with two independent variables (time variable t and space variable x) and one dependent variable u are derived under the transformation of the independent variables, by using the infinitesimal method. We also obtain the joint invariant equation for the above-mentioned equation under equivalence transformation. In fact, we prove a necessary and sufficient condition for a (1 + 1) parabolic equation to be reducible via a local equivalence transformation to the one-dimensional classical heat equation. This result provides practical criteria for reduction. Finally, examples of (1 + 1) Fokker–Planck equations from applications are given to verify the results obtained.

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1. Introduction

The Fokker–Planck (FP) equation was derived by Fokker [9] and Planck [22] for the distribution function describing Brownian motion. The Boltzmann equation, which was the first equation of motion derived for the distribution function of a dilute gas in position and velocity space, reduces to the FP equation in a system in which one particle is very large compared to the others. The FP equation is merely an equation of motion for the distribution function of fluctuation in a stochastic way. Mathematically, the FP equation is a linear second-order partial differential equation of parabolic type. Generally speaking, it is a diffusion equation with an additional first-order derivative with respect to the x term. In the mathematical literature, the FP equation is also called a forward Kolmogorov equation and describes the evolution of the transition probability density for a diffusion process.

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Besides kinetic theory, the FP equation models a wide variety of phenomena arising in diverse fields: probability theory (describing the Markov process, an FP equation appears as the master equation [16]), laser physics (the statistics of light may very well be treated by a FP equation [1]), electronics (supersonic conductors, Josephson tunnelling junction, relaxation of dipoles, second-order phase-locked loops [2, 8, 12, 25]), an optimal portfolio problem [3], etc.

In the case of one space variable, to which we restrict ourselves here just for the sake of simplicity, the FP equation is included in the parabolic equation

$$u_t = a(t, x)u_{xx} + b(t, x)u_x + c(t, x)u \tag{1}$$

where u is the unknown function, t and x are the time and space coordinates, respectively, and a, b and c are smooth functions of t and x, assumed to be given.

The general one-dimensional FP equation is of the form [10, 23]

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}[A(t, x)u] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[B(t, x)u] \tag{2}$$

where u is the probability density and A and B are the coefficients of drift and diffusion, respectively.

Lie [18] first systematically investigated the symmetry properties of (1). He obtained the complete group classification of parabolic equations of the form (1). As a matter of fact, Lie provided all the canonical forms of parabolic equations (1) which admit nontrivial point symmetries—he found four, namely, 0, 1, 3 and 5 symmetry cases apart from the trivial symmetries of homogeneity and superposition. Lie also developed methods of integrability of these equations.

Bluman and Cole [4, 6] used the Lie method to determine invariant solutions (also called similarity solutions) of the classical heat equation which was later classified in Olver [20], according to the optimal system of one-dimensional subalgebras. Bluman [5] also found the invariant solutions for (1 + 1) FP equations (1). Further, since an (1 + 1) FP equation is a particular case of (1), every one-dimensional FP equation with a five-dimensional group of Lie point symmetries can be locally transformed into the heat equation and vice versa. This is a special case of Lie's [18] result and is also contained in Bluman [7]. In other words, all Fokker–Planck equations with a five-dimensional Lie group of symmetries form an equivalence class of which the heat equation is the canonical member.

P S Laplace discovered the two semi-invariants $h = a_t + ab - c$, $k = b_x + ab - c$, known as Laplace invariants, in 1773 for the general linear hyperbolic second-order equation

$$u_{tx} + a(t, x)u_t + b(t, x)u_x + c(t, x)u = 0$$

with two independent variables t, x in his fundamental memoir [17] dedicated to the integration theory of linear partial differential equations. These two quantities h, k remain unchanged under the linear transformation of the dependent variable $\bar{u} = \sigma(t, x)u$. They were utilized for the group classification of the above differential equations [21] and to construct the Riemann function for the Cauchy initial value problem by the Lie group-theoretical method (see [13]).

Recently, it has come to our knowledge that Ibragimov [15] showed by the infinitesimal method that equation (1) has the second-order semi-invariants

$$a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}$$
$$K = \left(a_t - aa_{xx} + a_x^2\right)b - \tfrac{1}{2}b^2 a_x + (ab - aa_x)b_x - ab_t + a^2 b_{xx} - 2a^2 c_x \tag{3}$$

under the transformation of dependent variables. He called the semi-invariant K, a Laplace-type invariant. It is straightforward, having knowledge of K, to verify that K is indeed a semi-invariant for equation (1) under linear changes in the dependent variable by computing K for (1) and $\bar{K}$ for the transformed equation (1). Both are equal to each other.

In this paper we obtain the joint singular invariant equation of (1) under changes of both the dependent and independent variables, by the infinitesimal method. The outline of this paper is as follows. Section 2 focuses on obtaining the semi-invariants for the parabolic equation (1), i.e. the quantities remain unaltered under the transformation of independent variables only. Section 3 is devoted to deriving the joint singular invariant equation under equivalence transformations of equation (1). It is proved that a parabolic equation (1) is locally equivalent, via equivalence transformations of equation (1), to the classical heat equation if and only if the joint singular invariant equation is satisfied. We verify by physical examples that any parabolic equation (1) satisfying the joint singular invariant equation is equivalent to the heat equation. It is also pointed out that the Laplace-type invariant K is not enough to reduce a parabolic equation to the heat equation. Concluding remarks are made in section 5. The appendix contains details of the calculations of the results given in section 3.

2. Semi-invariants of the parabolic equations

In this section, we derive the semi-invariants for equation (1) under transformation of independent variables. We begin the section by stating some preliminaries.

Let us recall that an equivalence transformation of equation (1) is an invertible transformation $\bar{t} = \phi(t, x, u), \bar{x} = \varphi(t, x, u), \bar{u} = \psi(t, x, u)$ which preserves the order of the equation as well as the linearity and homogeneity. However, in general, the transformed equation has new coefficients $\bar{a}, \bar{b}, \bar{c}$.

It is also known that the set of all equivalence transformations of the equation (1) is an infinite group consisting of the linear transformations of the dependent variable

$$\bar{u} = \sigma(t, x)u \qquad \sigma(t, x) \neq 0 \tag{4}$$

and invertible changes of the independent variables of the form

$$\bar{t} = \phi(t) \qquad \bar{x} = \psi(t, x) \qquad \dot{\phi} \neq 0 \qquad \psi_x \neq 0 \tag{5}$$

where an overdot denotes differentiation with respect to t, $\phi(t)$, $\psi(t, x)$ and $\sigma(t, x)$ are arbitrary functions and $\bar{u}$ is a new dependent variable. Two equations of the form (1) are said to be (locally) equivalent if they can be related by an appropriate combination of the equivalence transformations (4), (5).

Let us consider the semi-invariants of equation (1) under transformation of the independent variables. These are combinations of the coefficients a, b, c and their derivatives which remain unaltered under the transformations of (5) alone. Let us define the generator of (1) by

$$X = \xi^1(t, x, u)\frac{\partial}{\partial t} + \xi^2(t, x, u)\frac{\partial}{\partial x} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \mu\frac{\partial}{\partial a} + \nu\frac{\partial}{\partial b} + \omega\frac{\partial}{\partial c}$$

where the functions $\mu = \mu(t, x, a, b, c)$, $\nu = (t, x, a, b, c)$ and $\omega = (t, x, a, b, c)$.

The symmetry operators are defined from an invariance condition

$$X(u_t - a(t, x)u_{xx} - b(t, x)u_x - c(t, x)u)|_{(1)} = 0$$

where the notation $|_{(1)}$ means evaluated on equation (1).

It gives us the determining equation

$$\zeta_t = a\zeta_{xx} + b\zeta_x + \mu u_{xx} + \nu u_x + \omega u \tag{6}$$

on equation (1). We find from the formulae given, for example, in [14, p 217] that

$$\begin{aligned}
\zeta_t &= -\left[u_t\left(\xi^1_t + u_t\xi^1_u\right) + u_x\left(\xi^2_t + u_t\xi^2_u\right)\right]\\
\zeta_x &= -\left[u_t\left(\xi^1_x + u_x\xi^1_u\right) + u_x\left(\xi^2_x + u_x\xi^2_u\right)\right]\\
\zeta_{xx} &= -\big[2u_{xx}\xi^2_x + u_x\xi^2_{xx} + 2u_x^2\xi^2_{xu} + 3u_xu_{xx}\xi^2_u + u_x^3\xi^2_{uu} + 2u_{tx}\xi^1_x + u_t\xi^1_{xx}\\
&\quad + 2u_tu_x\xi^1_{xu} + (u_tu_{xx} + 2u_xu_{tx})\xi^1_u + u_tu_x^2\xi^1_{uu}\big].
\end{aligned} \tag{7}$$

Here we consider the transformations of independent variables (not dependent variables) and the coefficients a, b and c of (1) induced by the independent variables.

Substituting equations (7) into equation (6) and replacing the term au_{xx} by $u_t - bu_x - cu$ (from (1)) and separating the coefficients of u_{tx}, u_tu_x, u_t, u_x and the remaining terms, we obtain the following equations:

$$\begin{aligned}
&\xi^1 = \xi^1(t) = p(t) \qquad \xi^2 = \xi^2(t, x) = q(t, x)\\
&\mu = 2aq_x - ap_t \qquad \nu = aq_{xx} + bq_x - bp_t - q_t \qquad \omega = -cp_t
\end{aligned} \tag{8}$$

where $p(t)$ and $q(t, x)$ are arbitrary given functions.

From equation (8), the infinitesimal transformations on t and x are

$$\bar{t} = t + \epsilon p(t) \qquad \bar{x} = x + \epsilon q(t, x)$$

where ϵ is a small parameter. Writing $\frac{\mathrm{d}\bar{t}}{\mathrm{d}t} = 1 + \epsilon\dot{p}(t) = \dot{\phi}(t)$, one arrives at the transformation

$$\bar{t} = \phi(t)$$

for t. In a similar manner, one can also obtain the transformation $\bar{x} = \psi(t, x)$ for x.

Equations (8) provide the generator for the infinitesimal changes in a, b and c:

$$X = (2aq_x - ap_t)\frac{\partial}{\partial a} + (aq_{xx} + bq_x - bp_t - q_t)\frac{\partial}{\partial b} + (-cp_t)\frac{\partial}{\partial c}. \tag{9}$$

The infinitesimal test $XJ = 0$ for the invariants $J(a, b, c)$ is of the form

$$(2aq_x - ap_t)\frac{\partial J}{\partial a} + (aq_{xx} + bq_x - bp_t - q_t)\frac{\partial J}{\partial b} + (-cp_t)\frac{\partial J}{\partial c} = 0.$$

Since p and q are arbitrary functions, there are in general no relations between their derivatives; the latter equation breaks up into the following three equations obtained by cancelling separately the terms with q_{xx}, q_x, p_t

$$\frac{\partial J}{\partial b} = 0 \qquad \frac{\partial J}{\partial a} = 0 \qquad \frac{\partial J}{\partial c} = 0.$$

Thus, there are in general no invariants $J(a, b, c)$ other than $J = \text{const}$. However, the restricted choice $q = q(t)$ results in a nonconstant $J = H\left(\frac{c}{a}\right)$ for an arbitrary function H. In a similar manner, one can obtain three other cases that result in nonconstant J.

Therefore, in general, one should look for, as the next step, the first-order differential semi-invariants, i.e. the semi-invariants of the form $J = J(a, a_t, a_x; b, b_t, b_x; c, c_t, c_x)$ via the once-extended generator (9)

$$\begin{aligned}
X &= (2aq_x - ap_t)\frac{\partial}{\partial a} + (aq_{xx} + bq_x - bp_t - q_t)\frac{\partial}{\partial b} + (-cp_t)\frac{\partial}{\partial c}\\
&\quad + (2a_tq_x - 2a_tp_t + 2aq_{tx} - ap_{tt} - a_xq_t)\frac{\partial}{\partial a_t} + (a_xq_x - a_xp_t + 2aq_{xx})\frac{\partial}{\partial a_x}\\
&\quad + (a_tq_{xx} + aq_{txx} + b_tq_x + bq_{tx} - q_{tt} - 2b_tp_t - bp_{tt} - b_xq_t)\frac{\partial}{\partial b_t}\\
&\quad + (a_xq_{xx} + aq_{xxx} + bq_{xx} - q_{tx} - b_xp_t)\frac{\partial}{\partial b_x}\\
&\quad + (-cp_{tt} - 2c_tp_t - c_xq_t)\frac{\partial}{\partial c_t} + (-c_xp_t - c_xq_x)\frac{\partial}{\partial c_x}.
\end{aligned}$$

The equation $XJ(a, a_t, a_x; b, b_t, b_x; c, c_t, c_x) = 0$, upon equating to zero first the terms with $q_{txx}, q_{xxx}, q_{tx}, p_{tt}$ and then those with q_{xx}, q_t yields

$$\frac{\partial J}{\partial b_t} = 0 \qquad \frac{\partial J}{\partial b_x} = 0 \qquad \frac{\partial J}{\partial a_t} = 0 \qquad \frac{\partial J}{\partial c_t} = 0$$

and

$$\frac{\partial J}{\partial a_x} = 0 \qquad \frac{\partial J}{\partial b} = 0$$

respectively. Hence, $J = J(a; c, c_x)$. Now the terms with q_x, p_t provide the following system of two equations:

$$2a\frac{\partial J}{\partial a} - c_x\frac{\partial J}{\partial c_x} = 0 \qquad a\frac{\partial J}{\partial a} + c\frac{\partial J}{\partial c} + c_x\frac{\partial J}{\partial c_x} = 0.$$

One can readily solve these two equations of the system to obtain $J = J(H_1)$, where $H_1 = \frac{ac_x^2}{c^3}$, provided that $c \neq 0$.

The first-order differential semi-invariant has the coefficients a, c and c_x, the derivative of c with respect to x, but does not contain the coefficient b and its derivatives. Therefore, the first-order differential invariant H_1 alone is not sufficient to show that two parabolic equations of the form (1) are equivalent under the linear transformations (5). So, we have to find the second-order differential semi-invariants.

Let us consider the second-order differential semi-invariant of the form

$$J(a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}; b, b_t, b_x, b_{tt}, b_{tx}, b_{xx}; c, c_t, c_x, c_{tt}, c_{tx}, c_{xx})$$

for the twice-extended generator (9). Following in the same manner as above, one first arrives at the equations

$$\begin{aligned} &\frac{\partial J}{\partial c_{tt}} = 0 \qquad \frac{\partial J}{\partial b_{tt}} = 0 \qquad \frac{\partial J}{\partial b_{tx}} = 0 \qquad \frac{\partial J}{\partial b_{xx}} = 0 \\ &\frac{\partial J}{\partial a_{tt}} = 0 \qquad \frac{\partial J}{\partial a_{tx}} = 0 \qquad \frac{\partial J}{\partial b_t} = 0. \end{aligned} \tag{10}$$

It follows from the equations (10) that $J = J(a, a_t, a_x, a_{xx}; b, b_x; c, c_t, c_x, c_{tx}, c_{xx})$. Now the equation $XJ = 0$ is reduced to the following system of seven equations:

$$\begin{aligned} &a\frac{\partial J}{\partial b_x} + 2a\frac{\partial J}{\partial a_{xx}} - c_x\frac{\partial J}{c_{xx}} = 0 \qquad 2a\frac{\partial J}{\partial a_t} - \frac{\partial J}{\partial b_x} - c_x\frac{\partial J}{\partial c_{tx}} = 0 \\ &a\frac{\partial J}{\partial a_t} + c\frac{\partial J}{\partial c_t} + c_x\frac{\partial J}{\partial c_{tx}} = 0 \qquad a\frac{\partial J}{\partial b} + 2a\frac{\partial J}{\partial a_x} + (b + a_x)\frac{\partial J}{\partial b_x} + 3a_x\frac{\partial J}{\partial a_{xx}} = 0 \\ &\frac{\partial J}{\partial b} + a_x\frac{\partial J}{\partial a_t} + c_x\frac{\partial J}{\partial c_t} + c_{xx}\frac{\partial J}{\partial c_{tx}} = 0 \\ &2a\frac{\partial J}{\partial a} + b\frac{\partial J}{\partial b} + 2a_t\frac{\partial J}{\partial a_t} + a_x\frac{\partial J}{\partial a_x} - c_x\frac{\partial J}{\partial c_x} - c_{tx}\frac{\partial J}{\partial c_{tx}} - 2c_{xx}\frac{\partial J}{\partial c_{xx}} = 0 \\ &a\frac{\partial J}{\partial a} + b\frac{\partial J}{\partial b} + c\frac{\partial J}{\partial c} + 2a_t\frac{\partial J}{\partial a_t} + a_x\frac{\partial J}{\partial a_x} + b_x\frac{\partial J}{\partial b_x} \\ &\qquad + 2c_t\frac{\partial J}{\partial c_t} + c_x\frac{\partial J}{\partial c_x} + a_{xx}\frac{\partial J}{\partial a_{xx}} + 2c_{tx}\frac{\partial J}{\partial c_{tx}} + c_{xx}\frac{\partial J}{\partial c_{xx}} = 0. \end{aligned} \tag{11}$$

If we use the theory of systems of homogeneous linear partial differential equations of the first order, one solves the system (11) to derive

$$J = J(H_1, H_2)$$

where H_1 is the same as we have found before, the first-order differential semi-invariant, and H_2 is the second-order differential semi-invariant given by

$$H_2 = \frac{2}{c^2}(c_t + ca_{xx} - 2cb_x - bc_x) + \frac{a_x c_x}{c^2} - 2\frac{a_t}{ac} + 2\frac{ba_x}{ac} - \frac{a_x^2}{ac} \tag{12}$$

provided that $a, c \neq 0$.

One can note that H_2 does not contain the terms c_{tx}, c_{xx}, even though J is obviously a function of these two and the others, as in the course of solving the system of seven equations we had to separate the equation.

The necessary condition for local equivalence of two parabolic equations (1) for $c \neq 0$ via transformations (5) of independent variables is that the semi-invariants H_1 and H_2 for the two equations be the same. The sufficient conditions are deduced by constructing transformations of the form (5).

Let us present some examples to illustrate the use of the above differential semi-invariants under the transformations (5).

Example 2.1. Consider the equation $u_t = xu_{xx} + \frac{1}{2}u_x + u$, with $a = x$, $b = \frac{1}{2}$, $c = 1$, which has $H_1 = H_2 = 0$. This equation is equivalent to the equation $u_t = u_{xx} + u$ (this equation also has $H_1 = H_2 = 0$) via the linear transformation $\bar{t} = t$, $\bar{x} = 2\sqrt{x}$, $\bar{u} = u$.

Example 2.2. Let us investigate the equation $u_t = tu_{xx} + t^2 u_x + tu$, with $a = t$, $b = t^2$, $c = t$, which has differential semi-invariants $H_1 = H_2 = 0$. The above equation can be reduced to the equation $u_t = u_{xx} + u$ by means of the linear transformation $\bar{t} = \frac{t^2}{2}$, $\bar{x} = \frac{t^3}{3} + x$, $\bar{u} = u$.

3. Singular invariant equation for parabolic equations

In this section, our objective is to find the joint differential invariant(s) for equation (1) under a change of the equivalence transformation. We know from [15] that $a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}$ and K are the non-zero differential semi-invariants under the linear transformation of dependent variable (4). From equation (8) of section 2, we look for an operator of the form

$$\begin{aligned} X = {} & \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial a} + \nu \frac{\partial}{\partial b} + \mu_t \frac{\partial}{\partial a_t} + \mu_x \frac{\partial}{\partial a_x} + \mu_{tt} \frac{\partial}{\partial a_{tt}} + \mu_{tx} \frac{\partial}{\partial a_{tx}} + \mu_{xx} \frac{\partial}{\partial a_{xx}} \\ & + \nu_t \frac{\partial}{\partial b_t} + \nu_x \frac{\partial}{\partial b_x} + \nu_{xx} \frac{\partial}{\partial b_{xx}} + \omega_x \frac{\partial}{\partial c_x}. \end{aligned}$$

Then it follows that

$$\begin{aligned} & Xa = \mu \qquad Xa_t = \mu_t \qquad Xa_x = \mu_x \\ & Xa_{tt} = \mu_{tt} \qquad Xa_{tx} = \mu_{tx} \qquad Xa_{xx} = \mu_{xx} \qquad XK = \Gamma \end{aligned} \tag{13}$$

where

$$\begin{aligned} \Gamma = {} & (-3K)p_t + \left(aa_{xx} - a_t - a_x^2\right) q_t + (3K)q_x + aq_{tt} + (aa_x)q_{tx} \\ & + (-2a^2)q_{txx} + (a^2 a_x)q_{xxx} + a^3 q_{xxxx} \end{aligned} \tag{14}$$

and p, q are defined as in section 2. Again we construct a generator from equations (13) and (14) in the space of the semi-invariants $a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}$ and K:

$$\begin{aligned} X = {} & X(a)\frac{\partial}{\partial a} + X(a_t)\frac{\partial}{\partial a_t} + X(a_x)\frac{\partial}{\partial a_x} + X(a_{tt})\frac{\partial}{\partial a_{tt}} + X(a_{tx})\frac{\partial}{\partial a_{tx}} \\ & + X(a_{xx})\frac{\partial}{\partial a_{xx}} + X(K)\frac{\partial}{\partial K} \end{aligned}$$

i.e.

$$X = \mu\frac{\partial}{\partial a} + \mu_t\frac{\partial}{\partial a_t} + \mu_x\frac{\partial}{\partial a_x} + \mu_{tt}\frac{\partial}{\partial a_{tt}} + \mu_{tx}\frac{\partial}{\partial a_{tx}} + \mu_{xx}\frac{\partial}{\partial a_{xx}} + \Gamma\frac{\partial}{\partial K} \tag{15}$$

where μ_t, μ_x, μ_{tt}, μ_{tx} and μ_{xx} are given in the appendix which also includes the derivation of the equations to obtain the joint differential invariants of equation (1).

Lemma 3.1. *There are no first-, second-, third- and fourth-order joint differential invariants for equation (1).*

Proof. See the appendix. □

If one solves the system of equations (A.3) of the appendix, using the theory of systems of homogeneous linear partial differential equations of the first order, one will obtain the singular invariant equation instead of a fifth-order differential invariant, namely

$$\begin{aligned}\lambda \equiv {} & 4a(2aK_{xx} - 5a_xK_x) - 12K\left(aa_{xx} - 2a_x^2\right) + a_x\left(4aa_{tt} - 9a_x^4\right) \\ & - 12a_ta_x\left(a_t + 2a_x^2\right) + 4a\left(3a_t + 6a_x^2 - 5aa_{xx}\right)a_{tx} \\ & + 2aa_x\left(16a_ta_{xx} - 12aa_{xx}^2 + 15a_x^2a_{xx}\right) - 4a^2a_{ttx} - 12a^2a_xa_{txx} \\ & - 4a^2a_{xxx}\left(2a_t - 4aa_{xx} + 3a_x^2\right) + 8a^3a_{txxx} - 4a^4a_{xxxxx} = 0.\end{aligned} \tag{16}$$

The heat equation $u_t = u_{xx}$ is of the form (1) with coefficients $a = 1$, $b = 0$, $c = 0$. It has the Laplace-type semi-invariant (3), $K = 0$, and moreover satisfies the singular invariant equation (16), i.e $\lambda = 0$.

It is straightforward to verify that the other canonical forms in Lie's classification

$$u_t = u_{xx} + \frac{A}{x^2}u \qquad u_t = u_{xx} + Z(x)u \qquad u_t = u_{xx} + Z(t,x)u$$

where $A \neq 0$ is a constant and Z is an arbitrary function, have $\lambda \neq 0$ for $Z_x \neq 0$.

We have the following result.

Theorem 3.1. *A necessary condition for the parabolic equation of the form (1) to be reducible to the heat equation is that the singular invariant equation (16), i.e.* $\lambda = 0$, *with respect to the group of general equivalence transformations (4) and (5), must be satisfied.*

At this stage the question arises whether $\lambda = 0$ is sufficient for a given equation of the form (1) to be transformable into the heat equation.

We now verify that the equations (4) and (5) are equivalence transformations of equation (1) and show the existence of such transformations once $\lambda = 0$ for reduction to the heat equation. One can proceed as follows. By the rules of derivatives, one obtains

$$D_t = \dot{\phi}(t)\bar{D}_{\bar{t}} + \psi_t\bar{D}_{\bar{x}} \qquad D_x = \psi_x\bar{D}_{\bar{x}}. \tag{17}$$

Application of (17) to (4) yields

$$\begin{aligned}&\bar{u}_{\bar{t}} = \frac{(\sigma u_t + u\sigma_t)}{\dot{\phi}} - \frac{\psi_t(\sigma u_x + u\sigma_x)}{\dot{\phi}\psi_x} \qquad \bar{u}_{\bar{x}} = \frac{1}{\psi_x}(\sigma u_x + u\sigma_x) \\ &\bar{u}_{\bar{x}\bar{x}} = \frac{1}{\psi_x^2}(\sigma u_{xx} + 2\sigma_xu_x + u\sigma_{xx}) - \frac{\psi_{xx}}{\psi_x^3}(\sigma u_x + u\sigma_x).\end{aligned} \tag{18}$$

Substituting the above equations (18) into the equation $\bar{u}_{\bar{t}} = \bar{a}\bar{u}_{\bar{x}\bar{x}} + \bar{b}\bar{u}_{\bar{x}} + \bar{c}\bar{u}$, one will arrive at the following equations

$$\begin{aligned}&\bar{a}(\bar{t},\bar{x}) = \frac{a\psi_x^2}{\dot{\phi}} \qquad \bar{b}(\bar{t},\bar{x}) = \frac{\psi_x}{\dot{\phi}}\left(b - 2a\frac{\sigma_x}{\sigma} + a\frac{\psi_{xx}}{\psi_x} - \frac{\psi_t}{\psi_x}\right) \\ &\bar{c}(\bar{t},\bar{x}) = \frac{1}{\dot{\phi}}\left(c - a\frac{\sigma_{xx}}{\sigma} - b\frac{\sigma_x}{\sigma} + 2a\frac{\sigma_x^2}{\sigma^2} + \frac{\sigma_t}{\sigma}\right).\end{aligned} \tag{19}$$

Thus the transformations (4), (5) conserve the linearity and homogeneity.

Suppose that equation (1) has the symmetry generator

$$X = \tau(t)\frac{\partial}{\partial t} + \xi(t,x)\frac{\partial}{\partial x} + \eta(t,x)u\frac{\partial}{\partial u} \quad \tau \neq 0.$$

This generator is transformable via canonical coordinates into the translation symmetry generator $X = \frac{\partial}{\partial \bar{t}}$. The transformations are of the form

$$\bar{t} = \bar{t}(t) \qquad \bar{x} = \bar{x}(t,x) \qquad \bar{u} = \omega(t,x)u.$$

It follows that

$$\bar{t} = \int_0^t \frac{\mathrm{d}s}{\tau(s)}.$$

Also we must have

$$\tau\frac{\partial \bar{x}}{\partial t} + \xi\frac{\partial \bar{x}}{\partial x} = 0 \qquad \tau\frac{\partial \omega}{\partial t} + \xi\frac{\partial \omega}{\partial x} = -\eta\omega.$$

Then, equation (1) will reduce to

$$\bar{u}_{\bar{t}} = \bar{a}(\bar{x})\bar{u}_{\bar{x}\bar{x}} + \bar{b}(\bar{x})\bar{u}_{\bar{x}} + \bar{c}(\bar{x})\bar{u}$$

which admits the symmetry

$$X = \frac{\partial}{\partial \bar{t}}.$$

We now consider the equation without the bars, namely

$$u_t = a(x)u_{xx} + b(x)u_x + c(x)u$$

which has $X = \frac{\partial}{\partial t}$ as the symmetry generator. We find the transformation that reduces this equation to the heat equation

$$\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}}$$

under (4) and (5): $\bar{t} = \phi(t), \bar{x} = \psi(t,x), \bar{u} = \sigma(t,x)u$. From the first equation of (19), we have

$$\psi = \pm\dot{\phi}^{1/2}\int \frac{\mathrm{d}x}{\sqrt{a(x)}} + \beta(t)$$

where $\beta(t)$ is, for the moment, an arbitrary function, provided $\dot{\phi}$ and a have the same sign.

Substituting the above equation into the second equation of (19), we obtain

$$\sigma = \nu(t)|a(x)|^{-1/4}\exp\left[\frac{1}{2}\int\frac{b(x)}{a(x)}\mathrm{d}x - \frac{1}{8}\frac{\ddot{\phi}}{\dot{\phi}}\left(\int\frac{\mathrm{d}x}{\sqrt{a(x)}}\right)^2 \mp \frac{1}{2}\frac{\dot{\beta}}{\dot{\phi}^{1/2}}\int\frac{\mathrm{d}x}{\sqrt{a(x)}}\right] \tag{20}$$

where $\nu(t)$ is as yet an arbitrary function.

Then inserting equation (20) into the third equation of (19), we arrive at the following equations:

$$\begin{aligned} -8C_1 &= \frac{\ddot{\phi}^2}{\dot{\phi}^2} - 2\left(\frac{\ddot{\phi}}{\dot{\phi}}\right)_t \\ -4C_2 &= \frac{\ddot{\phi}}{\dot{\phi}^{3/2}}\dot{\beta} - 2\left(\frac{\dot{\beta}}{\sqrt{\dot{\phi}}}\right)_t \\ -C_3 &= \frac{\ddot{\phi}}{4\dot{\phi}} + \frac{\dot{\beta}^2}{4\dot{\phi}} + \frac{\dot{\nu}}{\nu} \end{aligned} \tag{21}$$

where C_1, C_2 and C_3 are constants with C_1 given by

$$C_1 = a^{1/2}\big(a^{1/2}A_x\big)_x$$

where

$$A(x) = c - \frac{b_x}{2} + \frac{ba_x}{2a} + \frac{a_{xx}}{4} - \frac{3}{16}\frac{a_x^2}{a} - \frac{b^2}{4a}.$$

Also, C_1, C_2 and C_3 are constrained by the relation

$$A - C_3 - \frac{1}{2}C_1\left(\int \frac{\mathrm{d}x}{\sqrt{a}}\right)^2 - C_2\left(\int \frac{\mathrm{d}x}{\sqrt{a}}\right) = 0.$$

Equations (21) mean that the transformations (4) and (5) which reduce a given parabolic equation (1) with $\lambda = 0$ into the heat equation exist as one can obtain the solutions ϕ, β and ν after finding the function A and the constant C_1. The constants C_2 and C_3 are also not arbitrary and should be appropriately chosen utilizing the above constraining relation. This completes the proof for the sufficient condition for the existence of equivalence transformations (4) and (5).

It is opportune to remark that (21) provide the explicit transformations as given in (4) and (5) (once $\lambda = 0$ and A, C_1 are known as well as C_2 and C_3 are determined) for the reduction to the classical heat equation for time-independent parabolic equations (1). If $\lambda = 0$ for parabolic time-dependent equations the above theorem guarantees the existence of a point transformation that will reduce it to the heat equation. The proof of the theorem relies on knowledge of a symmetry which is reduced to time translations via canonical coordinates which in turn transform the parabolic equation to a time-independent parabolic equation. However, in practice, once $\lambda = 0$, one can find the transformation that will reduce the equation to the heat equation without knowledge of a symmetry by solving equations (19). That is, one will end up with

$$\begin{aligned}
\psi(t,x) &= \pm\dot{\phi}^{1/2}\int a(t,x)^{-1/2}\,\mathrm{d}x + \beta(t)\\
\sigma(t,x) &= \nu(t)|a(t,x)|^{-1/4}\exp\left\{\int \frac{b(t,x)}{2a(t,x)}\,\mathrm{d}x - \frac{1}{8}\frac{\ddot{\phi}}{\dot{\phi}}\left(\int \frac{\mathrm{d}x}{a(t,x)^{1/2}}\right)^2\right.\\
&\qquad \left.\mp\frac{1}{2}\frac{\dot{\beta}}{\dot{\phi}^{1/2}}\int \frac{\mathrm{d}x}{a(t,x)^{1/2}}\right\}
\end{aligned} \tag{22}$$

where σ satisfies

$$c(t,x) - a\left(\frac{\sigma_x}{\sigma}\right)_x + a\left(\frac{\sigma_x}{\sigma}\right)^2 - b\left(\frac{\sigma_x}{\sigma}\right) + \left(\frac{\sigma_t}{\sigma}\right) = 0 \tag{23}$$

which needs to be solved for β, ϕ and ν.

In view of the above, we can state the following general results.

Theorem 3.2. *A necessary and sufficient condition for a one-dimensional parabolic equation (1) (which includes the one-dimensional FP equation) to be locally equivalent to the classical one-dimensional heat equation is that the singular invariant equation (16) be satisfied, i.e.* $\lambda = 0$.

Corollary 3.1. *A one-dimensional parabolic equation (1) (which includes the one-dimensional FP equation) admits a nontrivial five-dimensional symmetry Lie algebra of point symmetries (in addition to the trivial homogeneity symmetry* $u\frac{\partial}{\partial u}$ *and infinite superposition symmetries* $\alpha\frac{\partial}{\partial u}$ *with* α *solving equation (1)) if and only if (16) holds, i.e.* $\lambda = 0$.

4. Applications

It is worthwhile to provide the following examples to show the worthiness of the singular invariant equation (16). All parabolic equations stated in the examples below have five nontrivial Lie point symmetries apart from the homogeneity and superposition trivial symmetries such as $\lambda = 0$. Most of the transformations below are constructable from (21). In example 4.8, the transformation is obtained via (22) and (23). It is worthwhile remarking that there is more than one transformation which can reduce a given parabolic equation (1) to the heat equation.

Example 4.1. Consider the equation $\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(gu) + \frac{D}{2}u_{xx}$ (see [10]) describing the diffusional process in a field of force of weight, which is of the form (1)

$$u_t = \frac{D}{2}u_{xx} + gu_x \tag{24}$$

with coefficients $a = \frac{D}{2}, b = g, c = 0$ and D, g are constants. This equation has $K = 0$ and satisfies the singular invariant equation (16). Therefore, it can be transformed into the heat equation by Lie's equivalence transformation. Using equations (21), the Lie equivalence transformation through which the above-stated parabolic equation (24) is reduced to the heat equation is

$$\bar{t} = -\frac{1}{t} \qquad \bar{x} = \sqrt{\frac{2}{D}}\frac{x}{t} - \frac{1}{t}$$
$$\bar{u} = u\sqrt{t}\left(\frac{D}{2}\right)^{-1/4} \exp\left\{\frac{g}{D}x + \frac{g^2}{2D}t + \frac{x^2}{2Dt} + \frac{1}{4t} - \sqrt{\frac{1}{2D}}\frac{x}{t}\right\}.$$

There is another transformation (see [24, 26]) through which equation (24) is reducible to the heat equation

$$\bar{t} = \frac{D}{2}t \qquad \bar{x} = x \qquad \bar{u} = u\exp\left(gx + \frac{g^2}{2}t\right).$$

Example 4.2. Consider the equation $\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}[(1 - x^2)^2 u]$ (see [19]) describing models in population genetics, which is of the form (1)

$$u_t = (1 - x^2)^2 u_{xx} - 8x(1 - x^2)u_x - 4(1 - 3x^2)u \tag{25}$$

with coefficients $a = (1 - x^2)^2, b = -8x(1 - x^2), c = -4(1 - 3x^2)$. Equation (25) has $K = 0$ and satisfies the singular invariant equation (16). Hence, equation (25) is reducible to the heat equation by means of the equivalence transformation

$$\bar{t} = -\frac{1}{t} \qquad \bar{x} = \frac{1}{2t}\ln\left(\frac{1 + x}{1 - x}\right) - \frac{1}{t}$$
$$\bar{u} = u\sqrt{t}(1 - x^2)^{\frac{3}{2}}\left(\frac{1 + x}{1 - x}\right)^{-\frac{1}{4t}} \exp\left\{t + \frac{1}{4t} + \frac{1}{16t}\ln^2\left(\frac{1 + x}{1 - x}\right)\right\}.$$

We can also get the following transformation (see [24, 26])

$$\bar{t} = t \qquad \bar{x} = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad \bar{u} = u(1 - x^2)^{3/2}\mathrm{e}^t.$$

Example 4.3. Let us consider the equation $\frac{\partial u}{\partial t} = \frac{\alpha}{2}\frac{\partial^2}{\partial x^2}[x^2(1-x)^2]$ (see [19]) also describing models in population genetics, which is in the form (1)

$$u_t = \frac{\alpha}{2}(x-x^2)^2 u_{xx} + 2\alpha(x-x^2)(1-2x)u_x + \alpha(1-6x+6x^2)u \qquad (26)$$

with coefficients $a = \frac{\alpha}{2}(x-x^2)^2$, $b = 2\alpha(x-x^2)(1-2x)$, $c = \alpha(1-6x+6x^2)$. Equation (26) has $K = 0$ and satisfies equation (16). Thus, equation (26) can be transformed into the heat equation by means of Lie's equivalence transformation

$$\bar{t} = -\frac{1}{t} \qquad \bar{x} = \sqrt{\frac{2}{\alpha}}\frac{1}{t}\ln\frac{x}{1-x} - \frac{1}{t}$$

$$\bar{u} = u\left(\frac{\alpha}{2}\right)^{-1/4}\sqrt{t}(x-x^2)^{\frac{3}{2}}\left(\frac{x}{1-x}\right)^{-\frac{1}{\sqrt{2\alpha t}}}\exp\left\{\frac{\alpha t}{8} + \frac{1}{4t} + \frac{1}{2\alpha t}\ln^2\frac{x}{1-x}\right\}.$$

It is also possible to obtain the following equivalence transformation (see [24, 26])

$$\bar{t} = \frac{\alpha}{2}t \qquad \bar{x} = \ln\frac{x}{1-x} \qquad \bar{u} = u(x-x^2)^{\frac{3}{2}}\exp\left\{\frac{\alpha}{8}t\right\}.$$

Example 4.4. Let us investigate another equation $\frac{\partial u}{\partial t} = \frac{\alpha}{2}\frac{\partial^2}{\partial x^2}[(x-c)^2u] + \beta\frac{\partial}{\partial x}[(x-c)u]$ (see [19]) which also describes models in population genetics. This equation can be expressed in the form of (1), i.e.

$$u_t = \frac{\alpha}{2}(x-c)^2u_{xx} + (2\alpha+\beta)(x-c)u_x + (\alpha+\beta)u \qquad (27)$$

with coefficients $a = \frac{\alpha}{2}(x-c)^2$, $b = (2\alpha+\beta)(x-c)$, $c = \alpha+\beta$ and α, β are constants. One can readily verify that equation (27) has $K = 0$ and satisfies the singular invariant equation (16). Therefore, equation (27) is reduced to the heat equation by means of the transformation

$$\bar{t} = -\frac{1}{t} \qquad \bar{x} = \sqrt{\frac{2}{\alpha}}\frac{\ln(x-c)}{t} - \frac{1}{t}$$

$$\bar{u} = \left(\frac{\alpha}{2}\right)^{-1/4}u\sqrt{t}(x-c)^{\left(\frac{3}{2}+\frac{\alpha}{\beta}+\frac{\ln(x-c)}{2\alpha t}-\sqrt{\frac{1}{2\alpha}}\frac{1}{t}\right)}\exp\left\{\frac{(\alpha+2\beta)^2}{8\alpha}t + \frac{1}{4t}\right\}.$$

One can also derive the following transformation (see [24, 26])

$$\bar{t} = t \qquad \bar{x} = \sqrt{2/\alpha}\ln(x-c) \qquad \bar{u} = u(x-c)^{\left(\frac{3}{2}+\frac{\beta}{\alpha}\right)}\exp\left\{\left(\frac{\beta^2}{2\alpha} + \frac{\beta}{2} + \frac{\alpha}{8}\right)t\right\}.$$

Example 4.5. Let us consider the Black–Scholes (see [3]) equation which is a primary differential equation used to determine the appropriate price or theoretical value of an option (an optimal portfolio problem) in the mathematics of finance

$$u_t + \tfrac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0$$

where A, B and C are constants. We can readily verify that the invariant $K = 0$ and the Black–Scholes equation satisfies equation (16). Therefore, the Black–Scholes equation can be transformed into the heat equation by the equivalence transformation

$$\bar{t} = \frac{1}{t} \qquad \bar{x} = \frac{\sqrt{2}}{At}\ln x - \frac{1}{t}$$

$$\bar{u} = \left(\frac{A^2}{2}\right)^{-1/4}u\sqrt{t}\,x^{\left(\frac{\mathcal{D}}{A^2}-\frac{1}{2A^2t}\ln x+\frac{1}{\sqrt{2}At}\right)}\exp\left\{-\frac{1}{4t} - Ct - \frac{\mathcal{D}^2}{2A^2}t\right\}$$

where $A \neq 0$, $\mathcal{D} = B - \frac{A^2}{2}$ (see also [11]). The Black–Scholes equation is also transformable into the heat equation by another equivalence transformation (see [11, p 396]) which can be constructed in the same way.

Example 4.6. Consider the equation $\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(kxu) + \frac{D}{2}\frac{\partial^2 u}{\partial x^2}$ (see [10]) which has the form of (1)

$$u_t = \frac{D}{2}u_{xx} + kxu_x + ku \tag{28}$$

with coefficients $a = \frac{D}{2}, b = kx, c = k$ and D, k are constants. This equation describes the Ornstein–Uhlenbeck process. We have $K = \frac{D}{2}k^2 x$ and equation (16) is satisfied. Equation (28) has the following nontrivial Lie point symmetries

$$X_1 = \mathrm{e}^{2kt}\frac{\partial}{\partial t} + kx\mathrm{e}^{2kt}\frac{\partial}{\partial x} - \frac{2}{D}k^2x^2u\mathrm{e}^{2kt}\frac{\partial}{\partial u} \qquad X_2 = \mathrm{e}^{-2kt}\frac{\partial}{\partial t} - kx\mathrm{e}^{-2kt}\frac{\partial}{\partial x} + ku\mathrm{e}^{-2kt}\frac{\partial}{\partial u}$$
$$X_3 = \mathrm{e}^{kt}\frac{\partial}{\partial x} - \frac{2}{D}kxu\mathrm{e}^{kt}\frac{\partial}{\partial u} \qquad X_4 = \frac{\partial}{\partial t} \qquad X_5 = \mathrm{e}^{-kt}\frac{\partial}{\partial x}.$$

Hence, the above equation (28) is reducible to the heat equation by means of the transformations (4) and (5)

$$\bar{t} = -\frac{1}{2k}\frac{\mathrm{e}^{-kt}}{\mathrm{e}^{kt} - \mathrm{e}^{-kt}} \qquad \bar{x} = \frac{1}{2k}\frac{\sqrt{\frac{8}{D}}kx - \mathrm{e}^{-kt}}{\mathrm{e}^{kt} - \mathrm{e}^{-kt}}$$
$$\bar{u} = \left(\frac{D}{2}\right)^{-1/4} u(\mathrm{e}^{kt} - \mathrm{e}^{-kt})^{1/2}\exp\left\{\frac{1}{\mathrm{e}^{kt} - \mathrm{e}^{-kt}}\left(\frac{k}{D}x^2\mathrm{e}^{kt} - \frac{x}{\sqrt{2D}} + \frac{1}{8k}\mathrm{e}^{-kt}\right) - \frac{kt}{2}\right\}.$$

The transformations also take the form (see [24, 26])

$$\bar{t} = \frac{D}{4k}\exp(2kt) \qquad \bar{x} = x\exp(kt) \qquad \bar{u} = u\exp(-kt).$$

Example 4.7. The equation of the Rayleigh-type process, $\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left(\gamma x - \frac{\mu}{x}\right)u\right] + \frac{\mu}{2}\frac{\partial^2 u}{\partial x^2}$ (see [24]), is of the form (1)

$$u_t = \frac{\mu}{2}u_{xx} + \left(\gamma x - \frac{\mu}{x}\right)u_x + \left(\gamma + \frac{\mu}{x^2}\right)u \tag{29}$$

with coefficients $a = \frac{\mu}{2}, b = \gamma x - \frac{\mu}{x}, c = \gamma + \frac{\mu}{x^2}$, where μ, γ are constants. We have $K = \frac{\mu\gamma^2}{2}x$ for this equation and equation (16) holds. Moreover, the nontrivial Lie point symmetries of this equation (29) are

$$X_1 = \mathrm{e}^{2\gamma t}\frac{\partial}{\partial t} + \gamma x\mathrm{e}^{2\gamma t}\frac{\partial}{\partial x} - 2\left(\frac{\gamma^2x^2}{\mu} + \gamma\right)u\mathrm{e}^{2\gamma t}\frac{\partial}{\partial u}$$
$$X_2 = \mathrm{e}^{-2\gamma t}\frac{\partial}{\partial t} - \gamma x\mathrm{e}^{-2\gamma t}\frac{\partial}{\partial x} - \gamma u\mathrm{e}^{-2\gamma t}\frac{\partial}{\partial u}$$
$$X_3 = \mathrm{e}^{\gamma t}\frac{\partial}{\partial t} - \left(2\frac{\gamma x}{\mu} + \frac{\mu}{x}\right)u\mathrm{e}^{\gamma t}\frac{\partial}{\partial u}$$
$$X_4 = \mathrm{e}^{-\gamma t}\frac{\partial}{\partial x} - \frac{\mu}{x}u\mathrm{e}^{-\gamma t}\frac{\partial}{\partial u} \qquad X_5 = \frac{\partial}{\partial t}.$$

Also the above equation (29) is transformable into the heat equation by means of the transformation (4), (5)

$$\bar{t} = -\frac{1}{2\gamma}\frac{\mathrm{e}^{-\gamma t}}{\mathrm{e}^{\gamma t} - \mathrm{e}^{-\gamma t}} \qquad x = \frac{1}{2\gamma}\frac{\sqrt{\frac{8}{\mu}}\gamma x - \mathrm{e}^{-\gamma t}}{\mathrm{e}^{\gamma t} - \mathrm{e}^{-\gamma t}}$$
$$\bar{u} = \left(\frac{\mu}{2}\right)^{-1/4}\frac{u}{x}(\mathrm{e}^{\gamma t} - \mathrm{e}^{-\gamma t})^{1/2}\exp\left\{\frac{1}{\mathrm{e}^{\gamma t} - \mathrm{e}^{-\gamma t}}\left(\frac{\gamma}{D}x^2\mathrm{e}^{\gamma t} - \frac{x}{\sqrt{2\mu}} + \frac{1}{8\gamma}\mathrm{e}^{-\gamma t}\right) - \frac{3\gamma t}{2}\right\}.$$

The transformation takes the form (see [24, 26])

$$\bar{t} = \frac{\mu}{4\gamma}\exp(2\gamma t) \qquad \bar{x} = x\exp(\gamma t) \qquad \bar{u} = \frac{u}{x}\exp(-2\gamma t).$$

Example 4.8. The FP equation

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}[a(t)x + b(t)]u + c(t)\frac{\partial^2 u}{\partial x^2} \tag{30}$$

(see [27]) is of the form (1) with coefficients $a = c(t), b = -(a(t)x + b(t)), c = -a(t)$. Equation (30) has $K = (a^2c + c\dot{a} - a\dot{c})x + abc - b\dot{c} + c\dot{b}$ and the coefficients of this equation satisfy the singular invariant equation (16). Therefore, equation (30) is reducible to the heat equation by means of the equivalence transformation [24]

$$\bar{t} = \gamma(t) \qquad \bar{x} = x\exp\{\alpha(t)\} + \beta(t) \qquad \bar{u} = u\exp\{-\alpha(t)\}$$

where

$$\alpha(t) = -\int_0^t a(s)\,\mathrm{d}s \qquad \beta(t) = -\int_0^t b(s)\exp\{\alpha(s)\}\,\mathrm{d}s$$
$$\gamma(t) = \int_0^t c(s)\exp\{2\alpha(s)\}\,\mathrm{d}s.$$

This transformation can be obtained from (22) and (23).

5. Concluding remarks

We have obtained two semi-invariants under the equivalence transformation (5) for the (1 + 1) parabolic equation by the infinitesimal method. We have also derived the joint invariant equation for the (1 + 1) parabolic equation and this equation is satisfied for all parabolic equations which have nontrivial five Lie point symmetries in addition to the trivial homogeneity symmetry $u\frac{\partial}{\partial u}$ and infinite superposition symmetries $\alpha\frac{\partial}{\partial u}$ with α solving equation (1). Moreover, we have proved sufficient conditions for reduction to the one-dimensional heat equation. That is, these equations can be reduced to the heat equation via appropriate Lie equivalence transformations (4), (5). Also, it has been verified by physical examples 4.6–4.8 that equations having five nontrivial Lie point symmetries cannot be reduced to the heat equation by merely using the semi-invariant K but also the singular invariant equation $\lambda = 0$ given in (16). Finally, several physical examples were given to verify the general results obtained.

If the singular invariant equation (16) is not satisfied, then the parabolic equation has three, one or zero nontrivial Lie point symmetries.

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Appendix

Here, we derive the equations for the derivation of the joint differential invariants of (1) and briefly state the method to obtain the singular invariant equation (16). We have (see section 3)

$$
\begin{aligned}
\mu_t &= (-2a_t)p_t + (-a)p_{tt} + (-a_x)q_t + (2a_t)q_x + (2a)q_{tx} \\
\mu_x &= (-a_x)p_t + (a_x)q_x + (2a)q_{xx} \\
\mu_{tt} &= (-3a_{tt})p_t + (-3a_t)p_{tt} + (-a)p_{ttt} + (-2a_{tx})q_t + (2a_{tt})q_x \\
&\qquad + (-a_x)q_{tt} + (4a_t)q_{tx} + (2a)q_{ttx} \\
\mu_{tx} &= (-2a_{tx})p_t + (-a_x)p_{tt} + (-a_{xx})q_t + (a_{tx})q_x + (a_x)q_{tx} + (2a_t)q_{xx} + (2a)q_{txx} \\
\mu_{xx} &= (-a_{xx})p_t + (3a_x)q_{xx} + (2a)q_{xxx}
\end{aligned}
$$

which are calculated by using the total differentiations

$$
\begin{aligned}
D_t &= \frac{\partial}{\partial t} + a_t\frac{\partial}{\partial a} + a_{tt}\frac{\partial}{\partial a_t} + a_{tx}\frac{\partial}{\partial a_x} + \cdots + b_t\frac{\partial}{\partial b} + b_{tt}\frac{\partial}{\partial b_t} + b_{tx}\frac{\partial}{\partial b_x} + \cdots \\
&\qquad + c_t\frac{\partial}{\partial c} + c_{tt}\frac{\partial}{\partial c_t} + c_{tx}\frac{\partial}{\partial c_x} + \cdots + K_t\frac{\partial}{\partial K} + K_{tt}\frac{\partial}{\partial K_t} + K_{tx}\frac{\partial}{\partial K_x} + \cdots \\
D_x &= \frac{\partial}{\partial x} + a_x\frac{\partial}{\partial a} + a_{xx}\frac{\partial}{\partial a_x} + a_{tx}\frac{\partial}{\partial a_t} + \cdots + b_x\frac{\partial}{\partial b} + b_{xx}\frac{\partial}{\partial b_x} + b_{tx}\frac{\partial}{\partial b_t} + \cdots \\
&\qquad + c_x\frac{\partial}{\partial c} + c_{xx}\frac{\partial}{\partial c_x} + c_{tx}\frac{\partial}{\partial c_t} + \cdots + K_x\frac{\partial}{\partial K} + K_{tx}\frac{\partial}{\partial K_t} + K_{xx}\frac{\partial}{\partial K_x} + \cdots
\end{aligned}
\tag{A.1}
$$

and μ, i.e. for example

$$
\begin{aligned}
\mu_t &= D_t(\mu) - a_t D_t(\xi^1) - a_x D_t(\xi^2) \\
&= D_t(2aq_x - ap_t) - a_t D_t(p) - a_x D_t(q).
\end{aligned}
$$

The other terms are calculated in a similar manner.

The infinitesimal test $XJ = 0$ for the invariants $J(a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}; K)$ is written as

$$
\begin{aligned}
&[2aq_x - ap_t]\frac{\partial J}{\partial a} + [(-2a_t)p_t + (-a)p_{tt} + (-a_x)q_t + (2a_t)q_x + (2a)q_{tx}]\frac{\partial J}{\partial a_t} \\
&\quad + [(-a_x)p_t + (a_x)q_x + (2a)q_{xx}]\frac{\partial J}{\partial a_x} + [(-3a_{tt})p_t + (-3a_t)p_{tt} + (-a)p_{ttt} \\
&\quad + (-2a_{tx})q_t + (2a_{tt})q_x + (-a_x)q_{tt} + (4a_t)q_{tx} + (2a)q_{ttx}]\frac{\partial J}{\partial a_{tt}} \\
&\quad + [(-2a_{tx})p_t + (-a_x)p_{tt} + (-a_{xx})q_t + (a_{tx})q_x + (a_x)q_{tx} + (2a_t)q_{xx} \\
&\quad + (2a)q_{txx}]\frac{\partial J}{\partial a_{tx}} + [(-a_{xx})p_t + (3a_x)q_{xx} + (2a)q_{xxx}]\frac{\partial J}{\partial a_{xx}} \\
&\quad + \left[(-3K)p_t + \left(aa_{xx} - a_t - a_x^2\right)q_t + (3K)q_x + aq_{tt} + (aa_x)q_{tx} + (-2a^2)q_{txx} \right. \\
&\quad + \left. \left(a^2 a_x\right)q_{xxx} + a^3 q_{xxxx}\right]\frac{\partial J}{\partial K} = 0.
\end{aligned}
$$

Equating to zero the coefficients of $q_{xxxx}, q_{ttx}, q_{txx}, q_{xxx}, q_{tx}, q_{xx}$ and q_x yields

$$
\begin{aligned}
&\frac{\partial J}{\partial K} = 0 \qquad \frac{\partial J}{\partial a_{tt}} = 0 \qquad \frac{\partial J}{\partial a_{tx}} = 0 \qquad \frac{\partial J}{\partial a_{xx}} = 0 \\
&\frac{\partial J}{\partial a_t} = 0 \qquad \frac{\partial J}{\partial a_x} = 0 \qquad \frac{\partial J}{\partial a} = 0.
\end{aligned}
$$

Hence, there are no invariants $J(a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}; K)$ other than $J = \text{const}$. Let us now consider the third-order differential invariants, i.e. those of the form $J(a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}, a_{ttt}, a_{ttx}, a_{txx}, a_{xxx}; K, K_t, K_x)$ for the once-extended generator (15)

$$
\begin{aligned}
X &= \mu\frac{\partial}{\partial a} + \mu_t\frac{\partial}{\partial a_t} + \mu_x\frac{\partial}{\partial a_x} + \mu_{tt}\frac{\partial}{\partial a_{tt}} + \mu_{tx}\frac{\partial}{\partial a_{tx}} + \mu_{xx}\frac{\partial}{\partial a_{xx}} + \Gamma\frac{\partial}{\partial K} \\
&\quad + \mu_{ttt}\frac{\partial}{\partial a_{ttt}} + \mu_{ttx}\frac{\partial}{\partial a_{ttx}} + \mu_{txx}\frac{\partial}{\partial a_{txx}} + \mu_{xxx}\frac{\partial}{\partial a_{xxx}} + \Gamma_t\frac{\partial}{\partial K_t} + \Gamma_x\frac{\partial}{\partial K_x}.
\end{aligned}
$$

One can easily calculate $\mu_{ttt}, \ldots, \mu_{xxx}$ as we have found earlier for μ_t. We have

$$\begin{aligned}
\mu_{ttt} &= (-4a_{ttt})p_t + (-6a_{tt})p_{tt} + (-4a_t)p_{ttt} + (-a)p_{tttt} + (-3a_{ttx})q_t \\
&\quad + (2a_{ttt})q_x + (-3a_{tx})q_{tt} + (6a_{tt})q_{tx} + (-a_x)q_{ttt} + (6a_t)q_{ttx} + (2a)q_{tttx} \\
\mu_{ttx} &= (-3a_{ttx})p_t + (-3a_{tx})p_{tt} + (-a_x)p_{ttt} + (-2a_{ttx})q_t + (a_{ttx})q_x \\
&\quad + (-a_{xx})q_{tt} + (2a_{tx})q_{tx} + (2a_{tt})q_{xx} + (a_x)q_{ttx} + (4a_t)q_{txx} + (2a)q_{ttxx} \\
\mu_{txx} &= (-2a_{txx})p_t + (-a_{xx})p_{tt} + (-a_{xxx})q_t + (3a_{tx})q_{xx} + (3a_x)q_{txx} + (2a_t)q_{xxx} + (2a)q_{txxx} \\
\mu_{xxx} &= (-a_{xxx})p_t + (-a_{xxx})q_x + (3a_{xx})q_{xx} + (5a_x)q_{xxx} + (2a)q_{xxxx} \\
\Gamma_t &= (-4K_t)p_t + (-3K)p_{tt} + (a_t a_{xx} + a a_{txx} - a_{tt} - 2a_x a_{tx} - K_x)q_t \\
&\quad + (3K_t)q_x + \left(a a_{xx} - a_x^2\right)q_{tt} + (3K + a_t a_x + a a_{tx})q_{tx} + (a)q_{ttt} \\
&\quad + (a a_x)q_{ttx} + (-4a a_t)q_{txx} + \left(2a a_t a_x + a^2 a_{tx}\right)q_{xxx} + (-2a^2)q_{ttxx} \\
&\quad + \left(a^2 a_x\right) q_{txxx} + \left(3a^2 a_t\right)q_{xxxx} + (a^3)q_{txxxx} \\
\Gamma_x &= (-3K_x)p_t + (a a_{xxx} - a_{tx} - a_x a_{xx})q_t + (2K_x)q_x + (a_x)q_{tt} \\
&\quad + (2a a_{xx} - a_t)q_{tx} + (3K)q_{xx} + (a)q_{ttx} + (-3a a_x)q_{txx} + \left(a^2 a_{xx} + 2a a_x^2\right)q_{xxx} \\
&\quad + (-2a^2)q_{txxx} + \left(4a^2 a_x\right)q_{xxxx} + (a^3)q_{xxxxx}.
\end{aligned}$$

The equation $XJ = 0$, upon first equating to zero the terms with $q_{txxxx}, q_{xxxxx}, q_{ttxx}, q_{txxx}, p_{tttt}$ and then those with $p_{ttt}, q_{xxxx}, q_{txx}, q_{xxx}, q_{tt}, q_{tx}, q_{xx}, q_x$, yields

$$\partial J/\partial K_t = 0 \qquad \partial J/\partial K_x = 0 \qquad \partial J/\partial a_{ttx} = 0 \qquad \partial J/\partial a_{txx} = 0 \qquad \partial J/\partial a_{ttt} = 0$$

and

$$\begin{aligned}
&\partial J/\partial a_{tt} = 0 \qquad \partial J/\partial a_{xxx} = 0 \qquad \partial J/\partial a_{tx} = 0 \qquad \partial J/\partial a_{xx} = 0 \\
&\partial J/\partial K = 0 \qquad \partial J/\partial a_t = 0 \qquad \partial J/\partial a_x = 0 \qquad \partial J/\partial a = 0.
\end{aligned}$$

Thus, there is no third-order joint differential invariant. So, one must seek, as the next step, the fourth-order joint differential invariants of the form

$$\begin{aligned}
&J(a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}, a_{ttt}, a_{ttx}, a_{txx}, a_{xxx}, a_{tttt}, a_{tttx}, a_{ttxx}, a_{txxx}, a_{xxxx}; \\
&\qquad K, K_t, K_x, K_{tt}, K_{tx}, K_{xx})
\end{aligned}$$

for the twice-extended generator (15). We have the following equations by proceeding in the same way as we have done before:

$$\begin{aligned}
\mu_{tttt} &= (-5a_{tttt})p_t + (-10a_{ttt})p_{tt} + (-10a_{tt})p_{ttt} + (-5a_t)p_{tttt} + (-a)p_{ttttt} \\
&\quad + (-4a_{tttx})q_t + (2a_{tttt})q_x + (-6a_{ttx})q_{tt} + (8a_{ttt})q_{tx} + (-4a_{tx})q_{ttt} \\
&\quad + (12a_{tt})q_{ttx} + (-a_x)q_{tttt} + (8a_t)q_{tttx} + (2a)q_{ttttx} \\
\mu_{tttx} &= (-4a_{tttx})p_t + (-6a_{ttx})p_{tt} + (-4a_{tx})p_{ttt} + (-a_x)p_{tttt} + (-3a_{ttxx})q_t \\
&\quad + (a_{tttx})q_x + (-3a_{txx})q_{tt} + (3a_{ttx})q_{tx} + (2a_{ttt})q_{xx} + (-a_{xx})q_{ttt} \\
&\quad + (3a_{tx})q_{ttx} + (6a_{tt})q_{txx} + (a_x)q_{tttx} + (6a_t)q_{ttxx} + (2a)q_{tttxx} \\
\mu_{ttxx} &= (-3a_{ttxx})p_t + (-3a_{txx})p_{tt} + (-a_{xx})p_{ttt} + (-2a_{txxx})q_t + (-a_{xxx})q_{tt} \\
&\quad + (3a_{ttx})q_{xx} + (6a_{tx})q_{txx} + (2a_{tt})q_{xxx} + (3a_x)q_{ttxx} + (4a_t)q_{txxx} + (2a)q_{ttxxx} \\
\mu_{txxx} &= (-2a_{txxx})p_t + (-a_{xxx})p_{tt} + (-a_{xxxx})q_t + (-a_{txxx})q_x + (-a_{xxx})q_{tx} \\
&\quad + (3a_{txx})q_{xx} + (3a_{xx})q_{txx} + (5a_{tx})q_{xxx} + (5a_x)q_{txxx} + (2a_t)q_{xxxx} + (2a)q_{txxxx} \\
\mu_{xxxx} &= (-a_{xxxx})p_t + (-2a_{xxxx})q_x + (2a_{xxx})q_{xx} + (8a_{xx})q_{xxx} + (7a_x)q_{xxxx} + (2a)q_{xxxxx}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{tt} = {} & (-5K_{tt})p_t + (-7K_t)p_{tt} + (-3K)p_{ttt} \\
& + \left(a_{tt}a_{xx} + 2a_t a_{txx} + a a_{ttxx} - a_{ttt} - 2a_{tx}^2 - 2a_x a_{ttx} - 2K_{tx}\right) q_t \\
& + (3K_{tt})q_x + (2a_t a_{xx} + 2a a_{txx} - a_{tt} - 4a_x a_{tx} - K_x)q_{tt} \\
& + (6K_t + a_{tt}a_x + 2a_t a_{tx} + a a_{ttx})q_{tx} + \left(a a_{xx} - a_x^2 + a_t\right)q_{ttt} \\
& + (3K + 2a_t a_x + 2a a_{tx})q_{ttx} + \left(-4a_t^2 - 4a a_{tt}\right)q_{txx} \\
& + \left(2a a_x a_{tt} + 4a a_t a_{tx} + 2a_t^2 a_x + a^2 a_{ttx}\right)q_{xxx} + (a)q_{tttt} + (a a_x)q_{tttx} \\
& + (-8a a_t)q_{ttxx} + \left(4a a_t a_x + 2a^2 a_{tx}\right)q_{txxx} + \left(6a a_t^2 + 3a^2 a_{tt}\right)q_{xxxx} \\
& + (-2a^2)q_{tttxx} + \left(a^2 a_x\right)q_{ttxxx} + \left(6a^2 a_t\right)q_{txxxx} + (a^3)q_{ttxxxx} \\
\Gamma_{tx} = {} & (-4K_{tx})p_t + (-3K_x)p_{tt} \\
& + (a_t a_{xxx} - a_{xx}a_{tx} - a_x a_{txx} + a a_{txxx} - a_{ttx} - K_{xx})q_t + (2K_{tx})q_x \\
& + (a a_{xxx} - a_x a_{xx})q_{tt} + (2K_x + 2a_t a_{xx} + 2a a_{txx} - a_{tt})q_{tx} + (3K_t)q_{xx} \\
& + (a_x)q_{ttt} + (2a a_{xx})q_{ttx} + (3K - 3a a_{tx} - 3a_t a_x)q_{txx} \\
& + \left(4a a_x a_{tx} + 2a a_t a_{xx} + 2a_t a_x^2 + a^2 a_{txx}\right) q_{xxx} + (a)q_{tttx} + (-3a a_x)q_{ttxx} \\
& + \left(2a a_x^2 + a^2 a_{xx} - 4a a_t\right) q_{txxx} + \left(4a^2 a_{tx} + 8a a_t a_x\right)q_{xxxx} + (-2a^2)q_{ttxxx} \\
& + \left(4a^2 a_x\right) q_{txxxx} + \left(3a^2 a_t\right) q_{xxxxx} + (a^3)q_{txxxxx} \\
\Gamma_{xx} = {} & (-3K_{xx})p_t + \left(a a_{xxxx} - a_{txx} - a_{xx}^2\right) q_t + (K_{xx})q_x + (a_{xx})q_{tt} \\
& + (3a a_{xxx} - 2a_{tx} + a_x a_{xx})q_{tx} + (5K_x)q_{xx} + (2a_x)q_{ttx} \\
& + \left(-a a_{xx} - a_t - 3a_x^2\right)q_{txx} + \left(3K + 2a_x^3 + 6a a_x a_{xx} + a^2 a_{xxx}\right)q_{xxx} \\
& + (a)q_{ttxx} + (-7a a_x)q_{txxx} + \left(10a a_x^2 + 5a^2 a_{xx}\right) q_{xxxx} + (-2a^2)q_{txxxx} \\
& + \left(7a^2 a_x\right) q_{xxxxx} + (a^3)q_{xxxxxx}.
\end{aligned}$$

In the same manner as above, one can find that there are no joint differential invariants of fourth order upon the insertion of the above equations into the twice-extended generator acting on J, $XJ = 0$. Therefore, there is no alternative other than extending the operator (15) three times. We shall look for the differential invariants of the form

$$\begin{aligned}
J(&a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}, a_{ttt}, a_{ttx}, a_{txx}, a_{xxx}, a_{tttt}, a_{tttx}, a_{ttxx}, a_{txxx}, a_{xxxx}, a_{ttttt}, a_{ttttx}, a_{tttxx}, \\
& a_{ttxxx}, a_{txxxx}, a_{xxxxx}; K, K_t, K_x, K_{tt}, K_{tx}, K_{xx}, K_{ttt}, K_{ttx}, K_{txx}, K_{xxx})
\end{aligned}$$

for the thrice-extended generator (15). We obtain the following equations

$$\begin{aligned}
\mu_{ttttt} = {} & (-6a_{tttt})p_t + (-15a_{ttt})p_{tt} + (-20a_{tt})p_{ttt} + (-15a_{tt})p_{tttt} \\
& + (-6a_t)p_{ttttt} + (-a)p_{tttttt} + (-5a_{tttx})q_t + (2a_{tttt})q_x + (-10a_{ttx})q_{tt} \\
& + (10a_{ttt})q_{tx} + (-10a_{ttx})q_{ttt} + (20a_{tt})q_{ttx} + (-5a_{tx})q_{tttt} \\
& + (20a_{tt})q_{tttx} + (-a_x)q_{ttttt} + (10a_t)q_{ttttx} + (2a)q_{tttttx} \\
\mu_{ttttx} = {} & (-5a_{tttx})p_t + (-10a_{ttx})p_{tt} + (-10a_{ttx})p_{ttt} + (-5a_{tx})p_{tttt} \\
& + (-a_x)p_{ttttt} + (-4a_{tttxx})q_t + (a_{ttttx})q_x + (-6a_{ttxx})q_{tt} + (4a_{tttx})q_{tx} \\
& + (2a_{tttt})q_{xx} + (-4a_{txx})q_{ttt} + (6a_{ttx})q_{ttx} + (8a_{ttt})q_{txx} + (-a_{xx})q_{tttt} \\
& + (4a_{tx})q_{tttx} + (12a_{tt})q_{ttxx} + (a_x)q_{ttttx} + (8a_t)q_{tttxx} + (2a)q_{ttttxx} \\
\mu_{tttxx} = {} & (-4a_{tttxx})p_t + (-6a_{ttxx})p_{tt} + (-4a_{txx})p_{ttt} + (-a_{xx})p_{tttt} + (-3a_{ttxxx})q_t \\
& + (-3a_{txxx})q_{tt} + (3a_{tttx})q_{xx} + (-a_{xxx})q_{ttt} + (9a_{ttx})q_{txx} + (2a_{ttt})q_{xxx} \\
& + (9a_{tx})q_{ttxx} + (6a_{tt})q_{txxx} + (3a_x)q_{tttxx} + (6a_t)q_{ttxxx} + (2a)q_{tttxxx}
\end{aligned}$$

$$\mu_{ttxxx} = (-3a_{ttxxx})p_t + (-3a_{txxx})p_{tt} + (-a_{xxx})p_{ttt} + (-2a_{txxxx})q_t + (-a_{ttxxx})q_x + (-a_{xxxx})q_{tt} + (-2a_{txxx})q_{tx} + (3a_{ttxx})q_{xx} + (-a_{xxx})q_{ttx} + (6a_{txx})q_{txx} + (5a_{ttx})q_{xxx} + (3a_{xx})q_{ttxx} + (10a_{tx})q_{txxx} + (2a_{tt})q_{xxxx} + (5a_x)q_{ttxxx} + (4a_t)q_{txxxx} + (2a)q_{ttxxx}$$

$$\mu_{txxxx} = (-2a_{txxxx})p_t + (-a_{xxxx})p_{tt} + (-a_{xxxxx})q_t + (-2a_{txxxx})q_x + (-2a_{xxxx})q_{tx} + (2a_{txxx})q_{xx} + (2a_{xxx})q_{txx} + (8a_{txx})q_{xxx} + (8a_{xx})q_{txxx} + (7a_{tx})q_{xxxx} + (7a_x)q_{txxxx} + (2a_t)q_{xxxxx} + (2a)q_{txxxxx}$$

$$\mu_{xxxxx} = (-a_{xxxxx})p_t + (-3a_{xxxxx})q_x + (10a_{xxx})q_{xxx} + (15a_{xx})q_{xxxx} + (9a_x)q_{xxxxx} + (2a)q_{xxxxxx}.$$

There is no need to calculate $\Gamma_{ttt}, \Gamma_{ttx}, \Gamma_{txx}, \Gamma_{xxx}$ as we have noted in earlier cases that J does not depend on these variables. Proceeding in the same manner, one first arrives at the equations

$$\frac{\partial J}{\partial K_{ttt}} = 0 \quad \frac{\partial J}{\partial K_{ttx}} = 0 \quad \frac{\partial J}{\partial K_{txx}} = 0 \quad \frac{\partial J}{\partial K_{xxx}} = 0 \quad \frac{\partial J}{\partial a_{ttttt}} = 0$$
$$\frac{\partial J}{\partial a_{ttttx}} = 0 \quad \frac{\partial J}{\partial a_{tttxx}} = 0 \quad \frac{\partial J}{\partial a_{ttxxx}} = 0 \quad \frac{\partial J}{\partial a_{txxxx}} = 0 \quad \frac{\partial J}{\partial a_{tttt}} = 0 \qquad \text{(A.2)}$$
$$\frac{\partial J}{\partial a_{tttx}} = 0 \quad \frac{\partial J}{\partial a_{ttxx}} = 0 \quad \frac{\partial J}{\partial a_{ttt}} = 0 \quad \frac{\partial J}{\partial K_{tt}} = 0 \quad \frac{\partial J}{\partial K_{tx}} = 0 \quad \frac{\partial J}{\partial K_{t}} = 0.$$

It follows from equations (A.2) that $J(a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}, a_{ttx}, a_{txx}, a_{xxx}, a_{txxx}, a_{xxxx}, a_{xxxxx}; K, K_x, K_{xx})$. Now the equation $XJ = 0$ reduces to the following system of 17 equations:

$$2\frac{\partial J}{\partial a_{xxxxx}} + a^2\frac{\partial J}{\partial K_{xx}} = 0 \qquad \frac{\partial J}{\partial a_{txxx}} - a\frac{\partial J}{\partial K_{xx}} = 0 \qquad 2\frac{\partial J}{\partial a_{ttx}} + \frac{\partial J}{\partial K_{xx}} = 0$$

$$a\frac{\partial J}{\partial a_{tt}} + a_x\frac{\partial J}{\partial a_{ttx}} = 0 \qquad 2a\frac{\partial J}{\partial a_{tt}} + a_x\frac{\partial J}{\partial a_{ttx}} + a\frac{\partial J}{\partial K_x} + 2a_x\frac{\partial J}{\partial K_{xx}} = 0$$

$$2a\frac{\partial J}{\partial a_{txx}} + 5a_x\frac{\partial J}{\partial a_{txxx}} - 2a^2\frac{\partial J}{\partial K_x} - 7aa_x\frac{\partial J}{\partial K_{xx}} = 0$$

$$2a\frac{\partial J}{\partial a_{xxxx}} + 9a_x\frac{\partial J}{\partial a_{xxxxx}} + a^3\frac{\partial J}{\partial K_x} + 7a^2a_x\frac{\partial J}{\partial K_{xx}} = 0$$

$$a_x\frac{\partial J}{\partial a_{tt}} + a_{xx}\frac{\partial J}{\partial a_{ttx}} - a\frac{\partial J}{\partial K} - a_x\frac{\partial J}{\partial K_x} - a_{xx}\frac{\partial J}{\partial K_{xx}} = 0$$

$$a\frac{\partial J}{\partial a_t} + 3a_t\frac{\partial J}{\partial a_{tt}} + a_x\frac{\partial J}{\partial a_{tx}} + 3a_{tx}\frac{\partial J}{\partial a_{ttx}} + a_{xx}\frac{\partial J}{\partial a_{txx}} + a_{xxx}\frac{\partial J}{\partial a_{txxx}} = 0$$

$$2a\frac{\partial J}{\partial a_{tx}} + 4a_t\frac{\partial J}{\partial a_{ttx}} + 3a_x\frac{\partial J}{\partial a_{txx}} + 3a_{xx}\frac{\partial J}{\partial a_{txxx}} - 2a^2\frac{\partial J}{\partial K} - 3aa_x\frac{\partial J}{\partial K_x} - \left(aa_{xx} + a_t + 3a_x^2\right)\frac{\partial J}{\partial K_{xx}} = 0$$

$$2a\frac{\partial J}{\partial a_{xxx}} + 2a_t\frac{\partial J}{\partial a_{txxx}} + 7a_x\frac{\partial J}{\partial a_{xxxx}} + 15a_{xx}\frac{\partial J}{\partial a_{xxxxx}} + a^3\frac{\partial J}{\partial K} + 4a^2a_x\frac{\partial J}{\partial K_x} + \left(5a^2a_{xx} + 10aa_x^2\right)\frac{\partial J}{\partial K_{xx}} = 0$$

$$
\begin{aligned}
&2a\frac{\partial J}{\partial a_t}+4a_t\frac{\partial J}{\partial a_{tt}}+a_x\frac{\partial J}{\partial a_{tx}}+2a_{tx}\frac{\partial J}{\partial a_{ttx}}-a_{xxx}\frac{\partial J}{\partial a_{txxx}}+aa_x\frac{\partial J}{\partial K}\\
&\qquad+(2aa_{xx}-a_t)\frac{\partial J}{\partial K_x}+(3aa_{xxx}-2a_{tx}+a_xa_{xx})\frac{\partial J}{\partial K_{xx}}=0\\
&2a\frac{\partial J}{\partial a_x}+2a_t\frac{\partial J}{\partial a_{tx}}+3a_x\frac{\partial J}{\partial a_{xx}}+2a_{tt}\frac{\partial J}{\partial a_{ttx}}+3a_{tx}\frac{\partial J}{\partial a_{txx}}+3a_{xx}\frac{\partial J}{\partial a_{xxx}}\\
&\qquad+3a_{txx}\frac{\partial J}{\partial a_{txxx}}+2a_{xxx}\frac{\partial J}{\partial a_{xxxx}}+3K\frac{\partial J}{\partial K_x}+5K_x\frac{\partial J}{\partial K_{xx}}=0\\
&2a\frac{\partial J}{\partial a_{xx}}+2a_t\frac{\partial J}{\partial a_{txx}}+5a_x\frac{\partial J}{\partial a_{xxx}}+5a_{tx}\frac{\partial J}{\partial a_{txxx}}+8a_{xx}\frac{\partial J}{\partial a_{xxxx}}+10a_{xxx}\frac{\partial J}{\partial a_{xxxxx}}\\
&\qquad+a^2a_x\frac{\partial J}{\partial K}+\left(2aa_x^2+a^2a_{xx}\right)\frac{\partial J}{\partial K_x}\\
&\qquad+\left(3K+2a_x^3+6aa_xa_{xx}+a^2a_{xxx}\right)\frac{\partial J}{\partial K_{xx}}=0\\
&a_x\frac{\partial J}{\partial a_t}+2a_{tx}\frac{\partial J}{\partial a_{tt}}+a_{xx}\frac{\partial J}{\partial a_{tx}}+2a_{txx}\frac{\partial J}{\partial a_{ttx}}+a_{xxx}\frac{\partial J}{\partial a_{txx}}+a_{xxxx}\frac{\partial J}{\partial a_{txxx}}\\
&\qquad-\left(aa_{xx}-a_t-a_x^2\right)\frac{\partial J}{\partial K}-(aa_{xxx}-a_{tx}-a_xa_{xx})\frac{\partial J}{\partial K_x}\\
&\qquad-\left(aa_{xxxx}-a_{txx}-a_{xx}^2\right)\frac{\partial J}{\partial K_{xx}}=0\\
&2a\frac{\partial J}{\partial a}+2a_t\frac{\partial J}{\partial a_t}+a_x\frac{\partial J}{\partial a_x}+2a_{tt}\frac{\partial J}{\partial a_{tt}}+a_{tx}\frac{\partial J}{\partial a_{tx}}+a_{ttx}\frac{\partial J}{\partial a_{ttx}}-a_{xxx}\frac{\partial J}{\partial a_{xxx}}-a_{txxx}\frac{\partial J}{\partial a_{txxx}}\\
&\qquad-2a_{xxxx}\frac{\partial J}{\partial a_{xxxx}}-3a_{xxxxx}\frac{\partial J}{\partial a_{xxxxx}}+3K\frac{\partial J}{\partial K}+2K_x\frac{\partial J}{\partial K_x}+K_{xx}\frac{\partial J}{\partial K_{xx}}=0\\
&a\frac{\partial J}{\partial a}+2a_t\frac{\partial J}{\partial a_t}+a_x\frac{\partial J}{\partial a_x}+3a_{tt}\frac{\partial J}{\partial a_{tt}}+2a_{tx}\frac{\partial J}{\partial a_{tx}}+a_{xx}\frac{\partial J}{\partial a_{xx}}+3a_{ttx}\frac{\partial J}{\partial a_{ttx}}\\
&\qquad+2a_{txx}\frac{\partial J}{\partial a_{txx}}+a_{xxx}\frac{\partial J}{\partial a_{xxx}}+2a_{txxx}\frac{\partial J}{\partial a_{txxx}}+a_{xxxx}\frac{\partial J}{\partial a_{xxxx}}+a_{xxxxx}\frac{\partial J}{\partial a_{xxxxx}}\\
&\qquad+3K\frac{\partial J}{\partial K}+3K_x\frac{\partial J}{\partial K_x}+3K_{xx}\frac{\partial J}{\partial K_{xx}}=0.
\end{aligned}
\tag{A.3}
$$

Writing the first equation of (A.3) in the characteristic form

$$\frac{\mathrm{d}a_{xxxxx}}{2}=\frac{\mathrm{d}K_{xx}}{a^2}=\frac{\mathrm{d}J}{0} \tag{A.4}$$

it follows that

$$J=J(A_1;a,a_t,a_x,a_{tt},a_{tx},a_{xx},a_{ttx},a_{txx},a_{xxx},a_{txxx},a_{xxxx};K,K_x)$$

where $A_1=2K_{xx}-a^2a_{xxxxx}$ is a solution of (A.4).

We write the second equation of (A.3) in the following way:

$$X=\frac{\partial}{\partial a_{txxx}}-a\frac{\partial}{\partial K_{xx}}.$$

Then, we have

$$XA_1=-2a \qquad Xa_{txxx}=1 \qquad Xa=0 \qquad Xa_t=0,\ldots \qquad XK_x=0.$$

Therefore, we have

$$-\frac{\mathrm{d}A_1}{2a}=\frac{\mathrm{d}a_{txxx}}{1}=\frac{\mathrm{d}J}{0}. \tag{A.5}$$

Thus

$$J = J(A_2; a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}, a_{ttx}, a_{txx}, a_{xxx}, a_{xxxx}; K, K_x)$$

where $A_2 = 2K_{xx} - a^2 a_{xxxxx} + 2aa_{txxx}$ is a solution of (A.5).

If one follows the same procedure up to the seventh equation of (A.3), one will end up with

$$J = J(B_1; a, a_t, a_x, a_{tx}, a_{xx}, a_{xxx}; K)$$

where

$$B_1 = a\left(2K_{xx} - a^2 a_{xxxxx} + 2aa_{txxx} - a_{ttx}\right) + a_x a_{tt} - 5a_x K_x - 3aa_x a_{txx}.$$

Proceeding in a similar way successively, one can obtain equation (16).

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